GEO111

Week #06: Basic geochemical box modelling and reservoir dynamics

I. A simple biological population model (using your most favorite bit of software – Excel)

Computer models are very important in all aspects of Earth, Ocean, and Atmospheric (and environmental) sciences. For example:

- A model of river flow may consider depth, flow rate, sediment load, etc.
- Models can be used to make predictions, for example, what effect would a reduction in flow rate resulting from river engineering have on the deposition of silt?
- This is much easier than going out and manipulating a river to find your answer. Duh!

You can use Excel to create simple mathematical models – the advantage of using Excel is that it is quicker and easier than learning some dreary computer programming language such as FORTRAN. As an example, we will consider a simple *population model.*

Modelling animal and plant populations using simple equations gives insights to the *population dynamics* (i.e. whether numbers remain stable, or go up and down slightly from year to year, or oscillate up and down wildly - almost to extinction one year and increasing to pest levels the next).

First consider the simple model:

$$N_{(t+1)} = \lambda . N_t \qquad (1)$$

This defines the number of individuals in the population that there will be in the future, based on the number in the current year.

N_t is the size of the population at time *t*.
N_{t+1} is the size of the population at time (*t*+1).
λ is the average number of offspring produced, per adult per year, less mortality.

Don't get put off by the '*N*'s and subscripts and things. All Equation 1 says is that the population size (number of individuals = *N*) at some time in the future (time = *t*+1) is equal to the population now (time = *t*) multiplied by some factor. This 'factor' is given the Greek letter λ. Because the units of λ are in per year (yr^{-1}), *t* represents the time in years since the start (of the model). The factor λ includes both gains due to the production of offspring and losses from the population due to snowboarding off of a cliff of some other way of dying.
So, we are simply asking; how many individuals will there be next year (time = *t* + 1)? The answer is; the same number as currently (this year, or time *t*), <u>minus</u> the fraction of the population who snowboard off of a cliff or die of old age, αN_t, <u>plus</u> the number of births in the population, which is also assumed proportional to the current number of individuals in the population, βN_t (we are ignoring the time between birth and being ready to produce offspring in this equation).

If there are N_t individuals at present (time = t), the number next year ($t = t+1$) is;

$$N_{t+1} = N_t + \beta \times N_t - \alpha \times N_t \qquad (1a)$$

Re-arranging this we get;

$$N_{t+1} = (1 + \beta - \alpha) \times N_t. \qquad (1b)$$

The only even faintly subversive thing that has happened in Eq. 1, is that all these factors have been included in the value of λ (= $1 + \beta - \alpha$).
Simple, eh? Mostly, that is about all there is to computer modelling. You know how much stuff (rabbits, snowboarders, cloud water droplets, whatever) there is currently (or at a specific point in time), and you want to predict how much there will be in the future, which you take to be one time-step away. You estimate the change in quantity (rabbits, snowboarders, cloud water droplets) that occurs over the course of one time-step, and add it to the current quantity. Because you then know the quantity at time t+1, you can go calculate the quantity at time t+2. Then, knowing the quantity at time t+2, you can calculate the quantity at time t+3. Then, etc etc etc. Zzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz
(Is this sounding like a mission for loops, yet?)

This model predicts that as long as $\lambda > 1$, the population will increase exponentially, year by year, without end. Think of bacterial cells dividing in a Petri dish. On each subsequent generation (or time step) there will be twice as many cells as there are currently (assuming that all the cells divide into two at the same rate and there is no mortality of cells). The value of λ in this example would be 2.
Most animal populations do not do this - instead they vary around some average level. This is because birth & death rates vary depending on the size of the population. For example:

- When the population is large, there may be little food to go round and the birth rate falls (or death rate increases).
- Or, when the population is very small, all individuals may have access to as much food as they can eat giving a high birth rate (or low death rate).

For the bacteria in a Petri dish, the population cannot go on expanding for ever – sooner or later the entire surface of the nutrient agar will be covered, leaving no free space for new cells to sit happily directly on the food. Later, the nutrients in the agar might start to become depleted. Toxic waste products might also start to build up, slowing down the rate of growth and cell doubling in the bacteria.

We can include a ‘density-dependence’ by modifying Equation 1, to give;

$$N_{(t+1)} = \frac{\lambda . N_t}{(1 + a.N_t)^b} \qquad (2)$$

There are two new parameters here:

b defines the strength of the density dependence and the dynamics of the population, and
a is a scaling factor.

STUFF TO DO:

- Take two columns in a new Excel Worksheet (any 2 will do) – one for the year and one for population size. Start with a population size of 100 at year $t = 1$. Assuming a certain value for the initial state of a numerical model is known as *initializing* that model. In our example, the model has been initialized with a population size of 100. Hell – you have too start somewhere, and maybe someone has gone out in the rain counting rabbits during one particular year, or sat at the base of a cliff counting falling snowboarders.
  - ∗ Enter the *parameter* values of the model; $\lambda = 50$, $a = 0.1$, $b = 0.1$ into three cells somewhere off to the side of the Worksheet. In three adjacent cells, enter in some text to remind yourself which parameter is which.
- You are going to use <u>fixed</u> addresses to reference these parameters in the model, because you will have to change the parameter values later and explore the results.
  - ∗ Recall that a cell address written in the format 'B2' or 'D9' is a <u>relative</u> address. When the formula is copied to another part of the spreadsheet the address changes so that it now refers to a different cell, but in such a way that the location of this 'target' is unchanged *relative* to the cell containing the address.
  - ∗ Sometimes we want to use a formula that refers to a cell containing a particular value in a way that does not change when we copy this formula to another cell. We do this with a *fixed* address: using the notation B2 instead of B2.
- Calculate the population size in each successive generation (2 to 100) using the model defined by **Equation 1**. Start by entering in the equation in the second row (i.e., time $t = 2$). Note that the number of individuals at time $t = 2$ depends on the value at time $t = 1$ (the contents of the cell immediately above). This is equivalent to stepping in time from one year to the next, calculating the new number of individuals based on the pervious year's value. Fill down the cell containing the formula for time $t = 2$ to create years 3 through 100. Set $\boldsymbol{\lambda} = \mathbf{2}$, so that you have something like a model for a bacterial population growing on your sandwich.
  - ∗ To check that you have put in the model correctly, the bacterial population should be 200 (= 2.0E+02) at time $t = 2$, and 51200 (= 5.120E+04) at time $t = 10$. If not then you are a complete muppet. Go and find out why and fix it.
- Plot a graph in Excel showing population change over 100 generations. There are rather a lot of individuals on your sandwich after only 100 generations. The question is; are you still feeling hungry?
  - ∗ You will find that it is difficult to set the *y*-axis scaling so that you can display much of the plot – either only the very first bit, or the very last bit. Try using a log scale for the *y*-axis and see if it helps display more of the information.
- Now calculate how the population size evolves from year to year using the model defined by **Equation 2**. Again, start by entering in the equation in the second row (i.e., time $t = 2$). You will probably need to use sets of parentheses '(…)' in order to structure your formula. Set λ back to a value 100. Again, start with a population size of 100 at year $t = 1$.
  - ∗ As check; your population should now be; 2.76×10^{17} at time $t = 33$, and 5.33×10^{17} at time $t = 40$.

- Investigate the effect of changing the value of parameter *b* on the dynamics of the population, keeping the values of the parameters λ and *a* constant. Increase the value of the parameter *b* and investigate how the dynamics change. Try values of *b* in the range 0.1 to 10. Note that you can position the graph in the same Worksheet so that you can view it at the same time as being able to edit the fixed reference cells containing the model parameters. Your choice of a linear or log *y*-axis scale – use the one that enables the most information to be presented and in the most useful way. Try and find the approximate range of values of *b* that give the following types of dynamic of the population;

  - **Monotonic Damping** (smooth approach to a stable equilibrium).
  - **Damped Oscillations** (oscillates to start with then dampens down to an equilibrium).
  - **Stable Limit Cycles** (regular pattern of peaks and troughs with the population repeatedly returning to exactly the same size).
  - **Chaos** (population bombs about all over the place with no regular pattern).

Don't spend too much time playing. I know how much fun you are having ;)

This is a genuine 24-carat time-dependent ('time-stepping') numerical model, although it doesn't seem that exciting because it is stuck in MS Excel. You are using successive cells (rows) in the Worksheet to store the value of the population at each particular time. You can see that the population value at each subsequent time (*t*+1) depends directly on the value at the previous time (*t*). Could you predict the population size far into the future (large *t*) analytically (i.e., write down an equation and solve it)? Only for the model given by Equation 1.
Note the use of fixed referencing in creating 'parameters', whose values can be easily updated, instantly affecting the entire model (i.e., all 100 rows) as well as updating the graphical display. Pretty useful eh?

Here you are using a numerical model to explore how a system behaves, and how sensitive the behaviour is to a critical parameter (*b* in this example). This sort of exploratory investigation can help you identify critical parameter values that have a profound (and maybe unexpected) effect – for instance, if parameter *b* related to something that was impacted by climate change, you might be able to determine the point in the future when climate change might make a population unstable. You might identify a certain population level as genetically 'viable' (anything below this being 'un-viable'). You might then be in a position to make recommendations about conserving this species. And all from just 'playing' around with a computer model!

Actually, some of the behaviour of population size in the model is probably not 'real'. We will see in the next Class that for certainly ranges of parameter value, the model is no longer numerically 'stable'. It is this that gives rise to some of the strange population size behaviour.